

MULTIPLICITY RESULTS FOR THE FRACTIONAL LAPLACIAN IN EXPANDED DOMAINS

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ABSTRACT. In this paper we establish the multiplicity of nontrivial weak solutions for the problem $(-\Delta)^\alpha u + u = h(u)$ in Ω_λ , $u = 0$ on $\partial\Omega_\lambda$, where $\Omega_\lambda = \lambda\Omega$, Ω is a smooth and bounded domain in \mathbb{R}^N , $N > 2\alpha$, λ is a positive parameter, $\alpha \in (0, 1)$, $(-\Delta)^\alpha$ is the fractional Laplacian and the nonlinear term $h(u)$ has a subcritical growth. We use minimax methods, the Ljusternick-Schnirelmann and Morse theories to get multiplicity result depending on the topology of Ω .

1. INTRODUCTION

This paper is concerned with the following problem

$$(1.1) \quad \begin{cases} (-\Delta)^\alpha u + u = h(u) & \text{in } \Omega_\lambda, \\ u > 0 & \\ u = 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

where $\Omega_\lambda = \lambda\Omega$, Ω is a smooth and bounded domain in \mathbb{R}^N , $N > 2\alpha$, λ is a positive parameter, $\alpha \in (0, 1)$, $(-\Delta)^\alpha$ is the fractional Laplacian operator, whose definition will be briefly recalled in the next section, and h satisfies suitable assumptions.

We are motivated in studying an equation involving the fractional Laplacian due to the great attention which has been given in these last years to problems involving fractional operators, both in \mathbb{R}^N and in bounded domains. Indeed these problems appear in many areas such as physics, economy, finance, optimization, obstacle problems, fractional diffusion and probabilistic. In particular, from a probability point of view, the fractional Laplacian is the infinitesimal generator of a Lévy process, see e.g. [11]. We also recall that a fractional Schrödinger equation has been derived by Laskin in the framework of the Fractional Quantum Mechanics. More information and applications are contained in some references such as [7, 18, 24, 25, 28].

On the other hand, in a beautiful series of papers, Benci, Cerami and Passaseo (see [8–10]) investigate the existence and multiplicity of positive solutions for equations of type $-\Delta u + \lambda u = u^{p-1}$ or $-\varepsilon \Delta u + u = f(u)$ in a bounded domain Ω with Dirichlet boundary conditions. In particular they develop a tool which allows to estimate the number of solutions depending on the “shape” of the domain (or of suitable “nearby” domains), whenever the parameters λ, ε or p tend to a suitable limit value. They use variational methods, and introduce suitable maps which permit to see “a photography” of Ω in a certain sublevel set of the energy functional related to the equation. Then the Ljusternick-Schnirelmann and Morse theory, based on the properties of the *category* and some Morse relations, are used in order to obtain the

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existence of multiple solutions. Later on, these general ideas are successfully applied also in other contests, such as the “zero mass” case in [27], Klein-Gordon and Schrödinger-Poisson type equations in [22, 23, 26], p -laplacian equations in [1, 2, 14–17], quasilinear equations in [3, 5], fractional Schrödinger equation in \mathbb{R}^N with a potential in [21], problems involving magnetic fields in expanding domains in [4, 6], among many others.

The aim of this paper is to show existence and multiplicity results of solutions for the fractional scalar field equation (1.1) in the expanding domain Ω_λ . We obtain the same type of results of the papers cited above: roughly speaking, for λ large enough the number of positive solutions is bounded below by topological invariants related to Ω_λ .

More precisely, let us assume that $h : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function verifying the following conditions:

- (H0) $h(s) = 0$ for $s \leq 0$;
- (H1) $h(s) = o(|s|)$ at the origin;
- (H2) $\lim_{|s| \rightarrow \infty} h(s)/|s|^{q-1} = 0$ for some $q \in (2, 2_\alpha^*)$ where $2_\alpha^* = 2N/(N - 2\alpha)$;
- (H3) there exists $\theta > 2$ such that $0 < \theta H(s) \leq sh(s)$ for all $s > 0$, where $H(s) = \int_0^s h(t) dt$;
- (H4) the function $s \mapsto h(s)/s$ is increasing for $s > 0$.

The typical function satisfying the above conditions is $h(s) = s^\mu$ for $s \geq 0$, with $1 < \mu < q - 1$, and $h(s) = 0$ for $s < 0$.

Our main results are the following.

Theorem 1.1. *Suppose that (H0)-(H4) hold. Then there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, problem (1.1) has at least $\text{cat } \Omega_\lambda$ weak solutions.*

For $Y \subset X$, we are denoting with $\text{cat}_X(Y)$ the Ljusternick-Schnirelmann category of X in Y , i.e. the least number of closed and contractible sets in X which cover Y . When $X = Y$ we just write $\text{cat}(X)$.

As usual, we get one more solution if the domain Ω_λ is not contractible, i.e.

Theorem 1.2. *Beside the assumptions of the previous theorem, assume that $\text{cat } \Omega_\lambda > 1$. Then there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, problem (1.1) has at least $\text{cat } \Omega_\lambda + 1$ weak solutions.*

If we replace (H1) and (H2) with slightly stronger conditions in order to deal with the second variation of the energy functional associated to problem (1.1), we can get a better result by using the Morse Theory. To this aim, let

- (H1') $h'(s) = o(|s|)$ at the origin;
- (H2') $\lim_{|s| \rightarrow \infty} h'(s)/|s|^{q-2} = 0$ for some $q \in (2, 2_\alpha^*)$.

Then we have

Theorem 1.3. *Suppose that (H0)-(H1')-(H2')-(H3)-(H4) hold. Then there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, the equation (1.1) has at least $2\mathcal{P}_1(\Omega_\lambda) - 1$ solutions, if counted with their multiplicity.*

Here $\mathcal{P}_1(\Omega_\lambda)$ denotes the Poincaré polynomial of Ω_λ evaluated in $t = 1$. This definition will be recalled later during the proof.

To prove our results we use variational methods. Indeed a functional on a Hilbert space can be defined in such a way that its critical points are exactly the solutions of (1.1). In this framework the assumption on h are quite natural in order to deal with Nehari manifolds,

Mountain Pass arguments and Palais-Smale condition. We recall that if I is a C^1 functional on a Hilbert manifold \mathcal{M} and $c \in \mathbb{R}$, a sequence $\{v_n\} \subset \mathcal{M}$ is said to be a *Palais-Smale sequence* for I at level c (briefly, a $(PS)_c$ sequence) if $I(v_n) \rightarrow c$ and $I'(v_n) \rightarrow 0$ in the tangent bundle. Furthermore, I is said to satisfy the *Palais-Smale condition* at level c if every $(PS)_c$ sequence has a convergent subsequence.

The functional related to our problem will turn out to be bounded from below on the “manifold solution” and verify the Palais-Smale condition at every level c , so the “photography method” of Benci and Cerami can be implemented and the classical Ljusternick-Schnirelmann and Morse theory can be used to estimate the number of critical points of the functional, that is, the number of solutions of (1.1).

In the proof of our results, we use some arguments that can be found in [1, 4, 5]. However due to the presence of the Fractional Laplacian, some estimates more refined are need, such as in Lemma 4.1, Propositions 4.2 and 4.4, for instance.

1.1. Notations. Let us introduce here few notations that will be used throughout the paper.

- $B_R(x)$ denotes the open ball in \mathbb{R}^N of radius R centered in x ; if $x = 0$ we write B_R .
- For $U \subset \mathbb{R}^N$, we denote with \mathcal{C}_U the half cylinder $U \times (0, +\infty) \subset \mathbb{R}^{N+1}$. In particular $\mathcal{C}_{\mathbb{R}^N} = \mathbb{R}^N \times (0, +\infty)$. Whenever an element of \mathcal{C}_U is written as (x, y) , it has always to be intended as $x \in U, y \in (0, +\infty)$.
- The lateral boundary of the cylinder is $\partial_L \mathcal{C}_U = \partial U \times [0, +\infty)$.

Other notations will be introduced along the paper as soon as we need. Finally, we will use C_1, C_2, \dots to denote suitable positive constants, whose exact value may change from line to line.

The plan of the paper is the following. In Section 2 we recall some facts on the fractional Laplacian and write the variational framework in which we will work. Section 3 is devoted to study the limit problem associated to our equation; in particular compactness results are proved and, *en passant*, also the existence of a ground state solution for (1.1). In Section 4 we introduce the barycenter map and its properties. Moreover a careful analysis of the ground states level in terms of λ is carried out. Finally, in Section 5 we give the proof of Theorem 1.1 and Theorem 1.2, and finally in Section 6, after recalling some facts and introducing some notations in classical Morse Theory, we prove Theorem 1.3.

2. PRELIMINARY RESULTS AND THE VARIATIONAL FRAMEWORK

In this section we start by introducing the functional framework necessary to apply variational methods and recover some known results about the different forms of definition of the fractional power of the laplacian with Dirichlet boundary condition.

Let us consider the half cylinder with base Ω_λ , i.e. $\mathcal{C}_{\Omega_\lambda}$ and let

$$H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) = \left\{ v \in H^1(\mathcal{C}_{\Omega_\lambda}); v = 0 \text{ on } \partial_L \mathcal{C} \text{ and } \|v\|_\alpha < \infty \right\},$$

where

$$\|v\|_\alpha = \left(k_\alpha^{-1} \int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha} |\nabla v|^2 dx dy + \int_{\Omega_\lambda} |tr_{\Omega_\lambda} v(x)|^2 dx \right)^{\frac{1}{2}},$$

$k_\alpha = 2^{1-2\alpha} \Gamma(1-\alpha)/\Gamma(\alpha)$, $\alpha \in (0, 1)$ and tr_{Ω_λ} is the trace operator given by $tr_{\Omega_\lambda} v = v(\cdot, 0)$ for $v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$. It is not difficult to see that $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ is a Hilbert space when

endowed with the norm $\|\cdot\|_\alpha$, which comes from the following inner product

$$\langle v, w \rangle_\alpha = \int_{\mathcal{C}_{\Omega_\lambda}} k_\alpha^{-1} y^{1-2\alpha} \nabla v \nabla w dx dy + \int_{\Omega_\lambda} v(x, 0) w(x, 0) dx.$$

Consider the following space

$$\mathcal{V}_0^\alpha(\Omega_\lambda) = \left\{ tr_{\Omega_\lambda} v; v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \right\}.$$

By [13, Proposition 2.1], there exists a trace operator from $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ into the fractional Sobolev space $H_0^\alpha(\Omega_\lambda)$. Then $\mathcal{V}_0^\alpha(\Omega_\lambda)$ is a subspace of the fractional Sobolev space $H^\alpha(\Omega_\lambda)$ and we consider it with the norm

$$\|u\|_{\mathcal{V}_0^\alpha(\Omega_\lambda)} = \left(\|u\|_{L^2(\Omega_\lambda)}^2 + \int_{\Omega_\lambda} \int_{\Omega_\lambda} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy \right)^{1/2}.$$

Moreover, by Trace Theorem and embeddings of fractional Sobolev spaces (see [19, Theorem 6.7] for instance) it follows that

$$\|tr_{\Omega_\lambda} v\|_{L^p(\Omega_\lambda)} \leq C \|v\|_\alpha, \quad \forall v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}),$$

where $p \in (1, 2_\alpha^*)$.

By [13, Proposition 2.1] it holds that

$$\mathcal{V}_0^\alpha(\Omega_\lambda) = \left\{ u \in L^2(\Omega_\lambda); u = \sum_{k=1}^{\infty} b_k \varphi_k \text{ such that } \sum_{k=1}^{\infty} b_k^2 \mu_k^\alpha < \infty \right\},$$

where hereafter (μ_k, φ_k) are the eigenpairs of $(-\Delta, H_0^1(\Omega_\lambda))$, μ_k repeated as much as its multiplicity.

Given $u \in C_0^\infty(\Omega_\lambda)$, with $u = \sum_{k=1}^{\infty} b_k \varphi_k$, we define the operator

$$(2.1) \quad (-\Delta)^\alpha u = \sum_{k=1}^{\infty} \mu_k^\alpha b_k \varphi_k$$

which extends by density on $\mathcal{V}_0^\alpha(\Omega_\lambda)$.

Instead of working with this definition, we can get a local realization of $(-\Delta)^\alpha$ by adding one more dimension. Indeed, as proved in [13, Section 2.1], for each $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ there exists a unique $\tilde{u} \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$, called the α -harmonic extension of u such that

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha} \nabla \tilde{u}) = 0 & \text{in } \mathcal{C}_{\Omega_\lambda} \\ \tilde{u} = 0 & \text{on } \partial_L \mathcal{C}_{\Omega_\lambda} \\ \tilde{u}(\cdot, 0) = u & \text{on } \Omega_\lambda. \end{cases}$$

Moreover, if $u = \sum_{k=1}^{\infty} b_k \varphi_k$ then

$$(2.2) \quad \tilde{u}(x, y) = \sum_{k=1}^{\infty} b_k \varphi_k(x) \psi(\mu_k^{1/2} y), \quad \forall (x, y) \in \mathcal{C}_{\Omega_\lambda},$$

where ψ solves the Bessel equation

$$(2.3) \quad \begin{cases} \psi''(s) + \frac{1-2\alpha}{s} \psi'(s) = \psi, & s > 0 \\ -\lim_{s \rightarrow 0^+} s^{1-2\alpha} \psi'(s) = k_\alpha \\ \psi(0) = 1. \end{cases}$$

Now, for a fixed $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ define the functional $\frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha} \Big|_{\Omega_\lambda \times \{0\}} \in \mathcal{V}_0^\alpha(\Omega_\lambda)^*$ by

$$\left\langle \frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot, 0), g \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} := \frac{1}{k_\alpha} \int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha} \nabla \tilde{u} \nabla \tilde{g} \, dx dy, \quad g \in \mathcal{V}_0^\alpha(\Omega_\lambda).$$

Integration by parts in the right hand side of the last equality explains the notation chosen to the functional, since

$$\left\langle \frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot, 0), g \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} = \left\langle \frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot, 0), g \right\rangle_{L^2(\Omega_\lambda)}$$

for all $g \in \mathcal{V}_0^\alpha(\Omega_\lambda)$, where

$$\frac{\partial \tilde{u}}{\partial y^\alpha}(x, 0) = - \lim_{y \rightarrow 0^+} y^{1-2\alpha} \frac{\partial \tilde{u}}{\partial y}(x, y) \quad \forall x \in \Omega_\lambda.$$

Then we can define an operator $A_\alpha : \mathcal{V}_0^\alpha(\Omega_\lambda) \rightarrow \mathcal{V}_0^\alpha(\Omega_\lambda)^*$ such that

$$A_\alpha u := \frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha} \Big|_{\Omega_\lambda \times \{0\}}.$$

Let us prove that the operators A_α and $(-\Delta)^\alpha$ defined in (2.1) are in fact the same, i.e., that for all $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$,

$$A_\alpha u = \sum_{k=1}^{\infty} \mu_k^\alpha b_k \varphi_k, \quad \text{where} \quad u = \sum_{k=1}^{\infty} b_k \varphi_k.$$

It is enough to show that for all $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$,

$$\left\langle \frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot, 0), \varphi_k \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} = \langle (-\Delta)^\alpha u, \varphi_k \rangle_{L^2(\Omega_\lambda)}, \quad \text{for all } k \in \mathbb{N}.$$

For $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ and $k \in \mathbb{N}$, by (2.2),

$$\tilde{u}(x, y) = \sum_{k=1}^{\infty} b_k \varphi_k(x) \psi(\mu_k^{1/2} y) \quad \text{and} \quad \widetilde{\varphi}_k(x, y) = \varphi_k(x) \psi(\mu_k^{1/2} y).$$

Now, integration by parts implies that, for $y > 0$,

$$\int_{\Omega_\lambda} y^{1-2\alpha} \nabla \tilde{u}(x, y) \nabla \widetilde{\varphi}_k(x, y) dx = y^{1-2\alpha} b_k \left(\mu_k \psi(\mu_k^{1/2} y)^2 + \psi'(\mu_k^{1/2} y)^2 \right).$$

Then, by (2.3)

$$\begin{aligned} \left\langle \frac{1}{k_\alpha} \frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot, 0), \varphi_k \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} &= \frac{1}{k_\alpha} \int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha} \nabla \tilde{u} \nabla \widetilde{\varphi}_k \, dx dy \\ &= \frac{1}{k_\alpha} \int_0^{+\infty} y^{1-2\alpha} b_k \left(\mu_k \psi(\mu_k^{1/2} y)^2 + \psi'(\mu_k^{1/2} y)^2 \right) dy \\ &= \frac{1}{k_\alpha} \lim_{\eta \rightarrow 0^+} y^{1-2\alpha} \mu_k^{1/2} b_k \psi'(\mu_k^{1/2} y) \psi(\mu_k^{1/2} y) \Big|_{y=\eta} \\ &= b_k \mu_k^\alpha \\ &= \langle (-\Delta)^\alpha u, \varphi_k \rangle_{L^2(\Omega_\lambda)}. \end{aligned}$$

Hence, in (1.1) we are going to understand $(-\Delta)^\alpha$ as A_α .

Let us pass to the definition of weak solution for problems involving the fractional Laplacian. We say that a function u is a solution of the linear problem

$$\begin{cases} (-\Delta)^\alpha u = f(x) & \text{in } \Omega_\lambda \\ u = 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

where $f \in \mathcal{V}_0^\alpha(\Omega_\lambda)^*$, if $u = \text{tr}_{\Omega_\lambda} v$, where $v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ is a solution of

$$\begin{cases} -\text{div}(y^{1-2\alpha} \nabla v) = 0 & \text{in } \mathcal{C}_{\Omega_\lambda} \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\Omega_\lambda} \\ \frac{1}{k_\alpha} \frac{\partial v}{\partial y^\alpha}(x, 0) = f(x) & x \in \Omega_\lambda. \end{cases}$$

Analogously, we say that $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ is a weak solution of (1.1) if $u = \text{tr}_{\Omega_\lambda} v$, where $v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ is a weak solution of

$$\begin{cases} -\text{div}(y^{1-2\alpha} \nabla v) = 0 & \text{in } \mathcal{C}_{\Omega_\lambda} \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\Omega_\lambda} \\ \frac{1}{k_\alpha} \frac{\partial v}{\partial y^\alpha} + v(x, 0) = h(v(x, 0)) & v > 0 \text{ in } \Omega_\lambda, \end{cases}$$

that is,

$$\int_{\mathcal{C}_{\Omega_\lambda}} k_\alpha^{-1} y^{1-2\alpha} \nabla v \nabla \psi dx dy + \int_{\Omega_\lambda} v(x, 0) \psi(x, 0) dx = \int_{\Omega_\lambda} h(v(x, 0)) \psi(x, 0) dx, \quad \forall \psi \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}).$$

As it is easy to see, this is equivalent to say that v is a critical point of the C^1 functional

$$I_\lambda(v) = \frac{k_\alpha^{-1}}{2} \int_{\mathcal{C}_\lambda} y^{1-2\alpha} |\nabla v|^2 dx dy + \frac{1}{2} \int_{\Omega_\lambda} |v(x, 0)|^2 dx - \int_{\Omega_\lambda} H(v(x, 0)) dx$$

in $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$.

It is not difficult to see that, in virtue of the assumptions on the nonlinearity h , the functional I_λ possesses a Mountain Pass Geometry: the mountain pass level will be denoted with $c(\Omega_\lambda) > 0$. We also define the *Nehari manifold* associated to I_λ by

$$(2.4) \quad \mathcal{M}_\lambda = \left\{ v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \setminus \{0\} : J_\lambda(v) = 0 \right\}$$

where

$$J_\lambda(v) := I'_\lambda(v)[v] = k_\alpha^{-1} \int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha} |\nabla v|^2 dx dy + \int_{\Omega_\lambda} |v(x, 0)|^2 dx - \int_{\Omega_\lambda} h(v(x, 0)) v(x, 0) dx.$$

We will need the following properties about \mathcal{M}_λ . They are standard, as well, and just based on the hypothesis made on the nonlinearity; for a proof one can follow e.g. [9, Lemma 2.2].

Lemma 2.1. *Let $\lambda > 0$. The following propositions hold true:*

1. *for every $u \in \mathcal{M}_\lambda$ it is $J'_\lambda(u)[u] < 0$;*
2. *\mathcal{M}_λ is a differentiable manifold radially diffeomorphic to the unit sphere S in $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ and bounded away from 0;*
3. *I_λ is bounded from below on \mathcal{M}_λ and*

$$(2.5) \quad 0 < c(\Omega_\lambda) = \inf_{\mathcal{M}_\lambda} I_\lambda = \inf_{u \neq 0} \sup_{t > 0} I_\lambda(tu).$$

In particular every nonzero function $v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ can be “projected” on \mathcal{M}_λ ; in other words we have an homeomorphism which just multiply a function by a positive constant (depending on the function)

$$(2.6) \quad v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \setminus \{0\} \longmapsto t_\lambda v \in \mathcal{M}_\lambda.$$

It is clear that \mathcal{M}_λ is a natural constraint for I_λ in the sense that

Corollary 2.2. *If v is a critical point of I_λ on \mathcal{M}_λ , then v is a nontrivial critical point of I_λ on $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$.*

Moreover, standard arguments show that the Palais-Smale sequences for I_λ restricted to \mathcal{M}_λ are Palais-Smale sequences for the free functional I_λ , and I_λ satisfies the Palais-Smale condition on \mathcal{M}_λ if and only if it satisfies the same condition on $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$.

Remark 1. *In the next sections we will use some auxiliary functionals: they differ from I_λ just for the domain on which these functionals are defined. In a similar way as in (2.4) we will define the Nehari manifolds related to these functionals and it is clear that analogous properties to all those stated on \mathcal{M}_λ hold, since they are essentially based on the structure of the functional, on the hypothesis made on the nonlinearity, and on how the Nehari manifold is defined. For this reason, the above cited properties will be used without any other comment through the paper.*

3. COMPACTNESS RESULTS AND EXISTENCE OF A GROUND STATE SOLUTION FOR I_λ

Now let us consider the half cylinder with base \mathbb{R}^N , $\mathcal{C}_{\mathbb{R}^N}$, and define

$$H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}) = \{v \in H^1(\mathcal{C}_{\mathbb{R}^N}) : \|v\|_{\mathcal{C}_{\mathbb{R}^N}} < \infty\},$$

where

$$\|v\|_{\mathcal{C}_{\mathbb{R}^N}} = \left(k_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla v|^2 dx dy + \int_{\mathbb{R}^N} |v(x, 0)|^2 dx \right)^{1/2}.$$

It is easy to see that $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ is a Hilbert space when endowed with the norm $\|\cdot\|_{\mathcal{C}_{\mathbb{R}^N}}$, which comes from the following inner product

$$\langle v, w \rangle_{\mathcal{C}_{\mathbb{R}^N}} = k_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} \nabla v \nabla w dx dy + \int_{\mathbb{R}^N} v(x, 0) w(x, 0) dx.$$

An important result we are going to use in this work is related with the existence of a positive ground state solution of the *limit problem*

$$(P_\infty) \quad (-\Delta)^\alpha u + u = h(u) \quad \text{in } \mathbb{R}^N,$$

i.e., the least energy solution for the functional

$$I_\infty(v) = \frac{k_\alpha^{-1}}{2} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla v|^2 dx dy + \int_{\mathbb{R}^N} |v(x, 0)|^2 dx - \int_{\mathbb{R}^N} H(v(x, 0)) dx.$$

It is standard to see that I_∞ has a Mountain Pass Geometry in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$, whose mountain pass level is denoted by $c(\mathbb{R}^N) > 0$. Moreover, we can define the Nehari manifold associated to I_∞

$$\mathcal{M}_\infty = \left\{ v \in H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}) : I'_\infty(v)[v] = 0 \right\}$$

and standard computations give

$$0 < c(\mathbb{R}^N) = \inf_{\mathcal{M}_\infty} I_\infty.$$

The theorem below states the existence of a ground state solution for (P_∞) , hence $c(\mathbb{R}^N)$ is achieved on a function of mountain pass type. The result is known in the literature (it can be obtained with similar arguments used in [1, Theorem 3.1]) but for completeness, and since it will be very useful for us, we prefer to give the proof.

Lemma 3.1. *Let $\{v_n\} \subset \mathcal{M}_\infty$ be a sequence satisfying $I_\infty(v_n) \rightarrow c(\mathbb{R}^N)$. Then, either*

a) $\{v_n\}$ has a strongly convergent subsequence in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$

or

b) there exists a sequence $\{x_n\} \subset \mathbb{R}^N$ such that, up to a subsequence, $|x_n| \rightarrow +\infty$ and $\bar{v}_n(x, y) := v_n(x - x_n, y)$ strongly converges in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$.

In particular, there exists a positive minimizer, hereafter denoted by \mathfrak{w}_∞ , for $c(\mathbb{R}^N)$.

Proof. By the Ekeland Variational Principle we can assume without loss of generality that $\{v_n\}$ is a $(PS)_{c(\mathbb{R}^N)}$ sequence for I_∞ on \mathcal{M}_∞ and then, by very known arguments, it follows that it is a $(PS)_{c(\mathbb{R}^N)}$ sequence to I_∞ on $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$. In a standard way one can prove that $\{v_n\}$ is bounded in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ and then, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$.

First case: $v \neq 0$. It is a simple matter to prove in this case $I'_\infty(v) = 0$. It follows from the Fatou Lemma, (H3) and the weak lower semicontinuity of the norm that

$$\begin{aligned} c(\mathbb{R}^N) &\leq I_\infty(v) \\ &= I_\infty(v) - \frac{1}{\theta} I'_\infty(v)[v] \\ &= \left(\frac{1}{2} - \frac{1}{\theta} \right) \|v\|_{\mathcal{C}_{\mathbb{R}^N}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v)v - H(v) \right) dx \\ &\leq \liminf_{n \rightarrow \infty} \left[\left(\frac{1}{2} - \frac{1}{\theta} \right) \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v_n)v_n - H(v_n) \right) dx \right] \\ &= c(\mathbb{R}^N), \end{aligned}$$

which implies that $I_\infty(v) = c(\mathbb{R}^N)$. Now let us prove that $v_n \rightarrow v$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ and for this it is enough to show that $\|v_n\|_{\mathcal{C}_{\mathbb{R}^N}} \rightarrow \|v\|_{\mathcal{C}_{\mathbb{R}^N}}$. By the weak semicontinuity of the norm it follows that

$$(3.1) \quad \|v\|_{\mathcal{C}_{\mathbb{R}^N}} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}}.$$

Supposing by contradiction that

$$\limsup_{n \rightarrow \infty} \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}} > \|v\|_{\mathcal{C}_{\mathbb{R}^N}},$$

Fatou Lemma implies that

$$\begin{aligned} c(\mathbb{R}^N) &= \limsup_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{\theta} \right) \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}}^2 + \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v_n)v_n - H(v_n) \right) dx \\ &> \left(\frac{1}{2} - \frac{1}{\theta} \right) \|v\|_{\mathcal{C}_{\mathbb{R}^N}}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v)v - H(v) \right) dx \\ &= c(\mathbb{R}^N), \end{aligned}$$

which is a contradiction. Then it follows that

$$\limsup_{n \rightarrow \infty} \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}} \leq \|v\|_{\mathcal{C}_{\mathbb{R}^N}}$$

and this together with (3.1) implies that $v_n \rightarrow v$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$.

Second case: $v = 0$. Then $\{v_n\}$ is not strongly convergent; indeed, if this were not the case, we would have a contradiction with the fact that $I_\infty(v_n) \rightarrow c(\mathbb{R}^N) > 0$. Hence there are $R, \gamma > 0$ and $\{x_n\} \subset \mathbb{R}^N$ such that, up to a subsequence

$$\int_{B_R(x_n)} |v_n(x, 0)|^2 dx \geq \gamma > 0$$

In fact, on the contrary, by the version of concentration compactness principle given in [20, Lemma 2.2], $v_n(\cdot, 0) \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_\alpha^*$. By this fact together with conditions (H0)-(H4), implies that

$$I_\infty(v_n) = \int_{\mathbb{R}^N} \left(\frac{1}{2} h(v_n(x, 0)) v_n(x, 0) - H(v_n(x, 0)) \right) dx + o_n(1) = o_n(1),$$

which contradicts again $I_\infty(v_n) \rightarrow c(\mathbb{R}^N) > 0$. Moreover, since $v = 0$, it follows that $|x_n| \rightarrow +\infty$. This follows because otherwise Sobolev embedding can be used to prove that $v \neq 0$. Since \mathbb{R}^N is invariant by translation, defining $\bar{v}_n(x, y) := v_n(x - x_n, y)$ we still have a $(PS)_{c(\mathbb{R}^N)}$ sequence for I_∞ , which is contained on \mathcal{M}_∞ and is bounded in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$. Then $\bar{v}_n \rightharpoonup \bar{v} \neq 0$ and hence, by the first case, $\bar{v}_n \rightarrow \bar{v}$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$, $I_\infty(\bar{v}) = c(\mathbb{R}^N)$ and \bar{v} is a ground state for I_∞ . \square

For what concerns our functional we have

Lemma 3.2. *For every $\lambda > 0$, the functional I_λ satisfies the Palais-Smale condition on $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$, and hence on \mathcal{M}_λ .*

Proof. Let $\{v_n\} \subset H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ be a sequence such that

$$I_\lambda(v_n) \rightarrow c \quad \text{and} \quad I'_\lambda(v_n) \rightarrow 0.$$

Thus, by (H3) we get

$$C_1 + o_n(1) \|v_n\|_\alpha \geq I_\lambda(v_n) - \frac{1}{\theta} I'_\lambda(v_n)[v_n] \geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \|v_n\|_\alpha^2,$$

which gives that $\{v_n\}$ is bounded in $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$. Then we may assume that, up to a subsequence, $v_n \rightharpoonup v$ in $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ and hence $tr_{\Omega_\lambda} v_n \rightarrow tr_{\Omega_\lambda} v$ in $L^s(\Omega_\lambda)$, with $2 \leq s < 2_\alpha^*$. Thus, since the nonlinearity h has subcritical growth, by standard calculations, we see that I_λ satisfies the Palais-Smale condition. \square

Then, taking into account that I_λ is bounded from below on \mathcal{M}_λ we have

Theorem 3.3. *For every $\lambda > 0$, $c(\Omega_\lambda)$ is achieved on a ground state solution denoted with $\mathfrak{w}_{\Omega_\lambda}$.*

4. THE BARYCENTER MAP AND BEHAVIOR OF THE MOUNTAIN PASS LEVELS

In this section, we study the behavior of some minimax levels with respect to the parameter λ . To do so, some preliminaries are in order.

Without any loss of generality, from now on we assume that $0 \in \Omega_\lambda$. Following [9], for $v \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ with compact support and such that $tr_{\Omega_\lambda} v^+ \not\equiv 0$, we define the *barycenter* or *center of mass* of v in the following way: first consider the “trivial” extension of

$v^+(\cdot, 0) = \text{tr}_{\Omega_\lambda} v^+$ to the whole \mathbb{R}^N (denoted by the same symbol) and then set

$$\beta(v) := \beta(v^+(\cdot, 0)) = \frac{\int_{\mathbb{R}^N} x |v^+(x, 0)|^2 dx}{\int_{\mathbb{R}^N} |v^+(x, 0)|^2 dx} \in \mathbb{R}^N.$$

For $R > r > 0$ let us denote by $A_{R,r}(\tilde{x})$ the open annulus in \mathbb{R}^N centered in \tilde{x}

$$A_{R,r}(\tilde{x}) = B_R(\tilde{x}) \setminus \overline{B_r}(\tilde{x}).$$

Define the functional on $H_{0,L}^1(\mathcal{C}_{A_{\lambda R, \lambda r}(\tilde{x})}, y^{1-\alpha})$

$$(4.1) \quad \widehat{I}_{\lambda, \tilde{x}}(v) = \frac{1}{2} \int_{\mathcal{C}_{A_{\lambda R, \lambda r}(\tilde{x})}} y^{1-2\alpha} |\nabla v|^2 dx dy + \frac{1}{2} \int_{A_{\lambda R, \lambda r}(\tilde{x})} |v(x, 0)|^2 dx \\ - \int_{A_{\lambda R, \lambda r}(\tilde{x})} H(v(x, 0)) dx,$$

and set

$$(4.2) \quad \widehat{\mathcal{M}}_{\lambda, \tilde{x}} = \left\{ v \in H_{0,L}^1(\mathcal{C}_{A_{\lambda R, \lambda r}(\tilde{x})}, y^{1-2\alpha}) \setminus \{0\}; \widehat{I}_{\lambda, \tilde{x}}(v)[v] = 0 \right\}$$

$$(4.3) \quad a(R, r, \lambda, \tilde{x}) = \inf \left\{ \widehat{I}_{\lambda, \tilde{x}}(v) : v \in \widehat{\mathcal{M}}_{\lambda, \tilde{x}} \text{ and } \beta(v) = \tilde{x} \right\}.$$

As is customary, when $\tilde{x} = 0$ we simply write \widehat{I}_λ , $\widehat{\mathcal{M}}_\lambda$ and $a(R, r, \lambda)$. We observe that the value $a(R, r, \lambda, \tilde{x})$ does not depend on the “center” \tilde{x} .

Since $\widehat{I}_{\lambda, \tilde{x}}$ has the Mountain Pass Geometry, is bounded from below on $\widehat{\mathcal{M}}_{\lambda, \tilde{x}}$ and satisfies the Palais-Smale condition, the infima $a(R, r, \lambda, \tilde{x})$ are obtained.

In the following we use a version of a maximum principle to the operator $(-\Delta)^\alpha$. Since we were not able to find in the literature the exact version of it which is necessary here, we prove it in the following result.

Lemma 4.1. *Let $\Gamma \subset \mathbb{R}^N$ be a smooth domain and $v \in H_{0,L}^1(\mathcal{C}_\Gamma, y^{1-2\alpha})$ such that*

$$(4.4) \quad \begin{cases} -\text{div}(y^{1-2\alpha} \nabla v) = 0 & \text{in } \mathcal{C}_\Gamma \\ v = 0 & \text{on } \partial_L \mathcal{C}_\Gamma \\ \frac{1}{k_\alpha} \frac{\partial v}{\partial y^\alpha}(x, 0) + v(x, 0) = f(x) & \text{on } \Gamma. \end{cases}$$

in the weak sense. If $f \geq 0$, then $v \geq 0$ in \mathcal{C}_Γ .

Proof. Since v satisfies (4.4), it follows that for all $\psi \in H_{0,L}^1(\mathcal{C}_\Gamma, y^{1-2\alpha})$ such that $\psi \geq 0$ in $\partial_L \mathcal{C}_\Gamma$, we have

$$k_\alpha^{-1} \int_{\mathcal{C}_\Gamma} y^{1-2\alpha} \nabla v \nabla \psi dx dy + \int_\Gamma v(x, 0) \psi(x, 0) dx = \int_\Gamma f(x) \psi(x, 0) dx.$$

If we take v^- (where $v = v^+ + v^-$) as a test function in the last expression we get

$$k_\alpha^{-1} \int_{\mathcal{C}_\Gamma} y^{1-2\alpha} |\nabla v^-|^2 dx dy + \int_\Gamma |v^-(x, 0)|^2 dx = \int_\Gamma f(x) v^- dx \leq 0.$$

But this implies that $v^- \equiv 0$ and then $v \geq 0$. □

The next result will be useful in future estimates with the barycenter map.

Proposition 4.2. *The number $a(R, r, \lambda)$ satisfies*

$$\liminf_{\lambda \rightarrow \infty} a(R, r, \lambda) > c(\mathbb{R}^N).$$

Proof. From the definition of $a(R, r, \lambda)$ and $c(\mathbb{R}^N)$, we get

$$a(R, r, \lambda) > c(\mathbb{R}^N).$$

Suppose by contradiction that there exist $\lambda_n \rightarrow \infty$ such that $a(R, r, \lambda_n) \rightarrow c(\mathbb{R}^N)$. Since $a(R, r, \lambda_n)$ is reached there exist $v_n \in \widehat{\mathcal{M}}_{\lambda_n}$ such that

$$\beta(v_n) = 0 \quad \text{and} \quad \widehat{I}_{\lambda}(v_n) = a(R, r, \lambda_n) \rightarrow c(\mathbb{R}^N).$$

Since $h \geq 0$, by (H0) and Lemma 4.1 it is $v_n \geq 0$ for all $n \in \mathbb{N}$. Moreover, since $v_n = 0$ on $\partial_L \mathcal{C}_{A_{\lambda_n R, \lambda_n r}}$, by considering the trivial extension on $\mathcal{C}_{\mathbb{R}^N} \setminus \mathcal{C}_{A_{\lambda_n R, \lambda_n r}}$ (which we denote with the same symbol) we obtain a function in $H_{0,L}^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$. Consequently,

$$v_n \rightharpoonup 0 \text{ in } H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}), \quad I_{\infty}(v_n) = a(R, r, \lambda_n) \rightarrow c(\mathbb{R}^N) \text{ and } v_n \in \mathcal{M}_{\infty}.$$

Recalling that $c(\mathbb{R}^N) > 0$, we have that $\{v_n\}$ is not strongly convergent. From Lemma 3.1, we get (recall $z = (x, y)$)

$$v_n(z) = w_n(z + z_n) + \mathfrak{w}_{\infty}(z + z_n)$$

where $\{w_n\} \subset H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ is a sequence converging strongly to 0, $\{z_n\} = \{(x_n, 0)\} \subset \mathbb{R}^{N+1}$ is such that $|x_n| \rightarrow \infty$ and $\mathfrak{w}_{\infty} \in H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ is a positive function verifying

$$I_{\infty}(\mathfrak{w}_{\infty}) = c(\mathbb{R}^N) \quad \text{and} \quad I'_{\infty}(\mathfrak{w}_{\infty}) = 0.$$

Since I_{∞} is rotationally invariant on functions of type $w(\cdot, 0)$, we can assume that

$$z_n = (x_n^1, 0, 0, \dots, 0) \quad \text{and} \quad x_n^1 < 0.$$

Now we set

$$M = \int_{\mathbb{R}^N} |\mathfrak{w}_{\infty}(x, 0)|^2 dx > 0.$$

Since $\|w_n\|_{\alpha} \rightarrow 0$, it follows that

$$\int_{B_{r\lambda_n/2}(x_n)} |w_n(x + x_n, 0) + \mathfrak{w}_{\infty}(x + x_n, 0)|^2 dx \rightarrow M,$$

from which we obtain

$$\int_{\Theta_n} |v_n(x, 0)|^2 dx \rightarrow M, \quad \text{where} \quad \Theta_n = B_{r\lambda_n/2}(x_n) \cap A_{\lambda_n R, \lambda_n r}$$

and hence

$$(4.5) \quad \int_{\Upsilon_n} |v_n(x, 0)|^2 dx \rightarrow 0, \quad \text{where} \quad \Upsilon_n = A_{\lambda_n R, \lambda_n r} \setminus B_{\lambda_n r/2}(x_n).$$

Since $\beta(v_n) = 0$, we get

$$0 = \int_{A_{\lambda_n R, \lambda_n r}} x^1 |v_n(x, 0)|^2 dx = \int_{\Theta_n} x^1 |v_n(x, 0)|^2 dx + \int_{\Upsilon_n} x^1 |v_n(x, 0)|^2 dx.$$

Thus,

$$-\frac{r\lambda_n}{2}(M + o_n(1)) + R\lambda_n \int_{\Upsilon_n} |v_n(x, 0)|^2 dx \geq 0$$

with $o_n(1) \rightarrow 0$. Then,

$$\int_{\Upsilon_n} |v_n(x, 0)|^2 dx \geq \frac{rM}{2R} - o_n(1)$$

which contradicts (4.5). \square

The other auxiliary functional we need is $I_{B_\xi} : H_{0,L}^1(\mathcal{C}_{B_\xi}, y^{1-2\alpha}) \rightarrow \mathbb{R}$, where $\xi > 0$, given by

$$(4.6) \quad I_{B_\xi}(v) = \frac{k_\alpha^{-1}}{2} \int_{\mathcal{C}_{B_\xi}} y^{1-2\alpha} |\nabla v|^2 dx dy + \frac{1}{2} \int_{B_\xi} |v(x, 0)|^2 dx - \int_{B_\xi} H(v(x, 0)) dx.$$

This functional has a Mountain Pass Geometry and we denote with $c(B_\xi)$ the mountain pass level. If

$$\mathcal{M}_{B_\xi} = \left\{ v \in H_{0,L}^1(\mathcal{C}_{B_\xi}, y^{1-2\alpha}) \setminus \{0\} : I'_{B_\xi}(v)[v] = 0 \right\}$$

denotes the Nehari manifold associated to I_{B_ξ} , then, as usual,

$$(4.7) \quad c(B_\xi) = \inf_{v \in \mathcal{M}_{B_\xi}} I_{B_\xi}(v).$$

Arguing as in Theorem 3.3 and using Schwartz symmetrization techniques, we get

Proposition 4.3. *The functional I_{B_ξ} defined in (4.6) satisfies the (PS) condition on \mathcal{M}_{B_ξ} . In particular there exists a ground state solution $\mathbf{w}_{B_\xi} \in \mathcal{M}_{B_\xi}$ and $\mathbf{w}_{B_\xi}(\cdot, 0)$ is radially symmetric with respect to the origin.*

Proposition 4.4. *The numbers $c(\Omega_\lambda)$ and $c(B_\xi)$, defined respectively in (2.5) and (4.7), verify the limits*

$$\lim_{\lambda \rightarrow \infty} c(\Omega_\lambda) = c(\mathbb{R}^N) \quad \text{and} \quad \lim_{\xi \rightarrow \infty} c(B_\xi) = c(\mathbb{R}^N).$$

Proof. Here we will just prove the first limit, since the second one follows from the same kind of arguments. Let Φ be a function in $C^\infty(\mathcal{C}_{\mathbb{R}^N}; [0, 1])$ such that

$$\Phi(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \mathcal{C}_{B_1} \\ 0 & \text{if } (x, y) \in \mathcal{C}_{\mathbb{R}^N \setminus B_2}. \end{cases}$$

For each $R > 0$, let us consider the rescaled function $\Phi_R(x, y) = \Phi(x/R, y)$ and set $w_R(x, y) = \Phi_R(x, y) \mathbf{w}_\infty(x, y)$, where \mathbf{w}_∞ is the ground state of the limit problem given in Lemma 3.1, hence $I_\infty(\mathbf{w}_\infty) = c(\mathbb{R}^N)$ and $I'_\infty(\mathbf{w}_\infty) = 0$. Observe that

$$(4.8) \quad w_R \rightarrow \mathbf{w}_\infty \quad \text{in} \quad H_{0,L}^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}) \quad \text{as} \quad R \rightarrow +\infty.$$

Since $0 \in \Omega_\lambda$, there exists $\bar{\lambda} > 0$ such that $B_{2R} \subset \Omega_\lambda$ for $\lambda \geq \bar{\lambda}$. Let $t_R > 0$ such that

$$I_\lambda(t_R w_R) = \max_{t \geq 0} I_\lambda(t w_R) = \max_{t \geq 0} I_\infty(t w_R).$$

Thus $I'_\lambda(t_R w_R)[t_R w_R] = 0$, i.e. $t_R w_R \in \mathcal{M}_\lambda$. Then

$$c(\Omega_\lambda) \leq I_\lambda(t_R w_R) = I_\infty(t_R w_R) \quad \text{for all } \lambda \geq \bar{\lambda}.$$

Since R is independent on λ , so is t_R . Hence, by taking the limit when $\lambda \rightarrow \infty$, we obtain

$$(4.9) \quad \limsup_{\lambda \rightarrow \infty} c(\Omega_\lambda) \leq I_\infty(t_R w_R).$$

Claim: we have $\lim_{R \rightarrow \infty} t_R = 1$.

Since $t_R w_R \in \mathcal{M}_\lambda$, we get

$$\begin{aligned} \|w_R\|_\alpha^2 &= \kappa_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla w_R|^2 dx dy + \int_{\mathbb{R}^N} |w_R(x, 0)|^2 dx \\ &= \int_{\mathbb{R}^N} h(t_R w_R(x, 0)) t_R^{-1} w_R(x, 0) dx \\ &> \int_{B_1} h(t_R m) t_R^{-1} m dx, \end{aligned}$$

where $m = \min_{|x| \leq 1} w_R(x, 0) > 0$ by the Strong Maximum Principle (see [12, Remark 4.2]). It follows that $\{t_R\}$ has to be bounded, otherwise by (H4) we deduce $\|w_R\|_\alpha^2 \rightarrow +\infty$, against (4.8).

Moreover, if there exists $R_n \rightarrow \infty$ with $t_{R_n} \rightarrow 0$, by (H1) and (H2)

$$\begin{aligned} \|w_{R_n}\|_{\mathcal{C}_{\mathbb{R}^N}}^2 &= \int_{\mathbb{R}^N} h(t_{R_n} w_{R_n}(x, 0)) t_{R_n}^{-1} w_{R_n}(x, 0) dx \\ &\leq C_1 t_{R_n} \int_{\mathbb{R}^N} |w_{R_n}(x, 0)|^2 dx + C_2 t_{R_n}^{q-1} \int_{\mathbb{R}^N} |w_{R_n}(x, 0)|^q dx \rightarrow 0 \end{aligned}$$

which again contradicts (4.8). This implies that $t_R \not\rightarrow 0$. Thus, we can assume that $t_R \rightarrow t_0 > 0$ for $R \rightarrow +\infty$ and consequently

$$\kappa_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla \mathbf{w}_\infty|^2 dx dy + \int_{\mathbb{R}^N} |\mathbf{w}_\infty(x, 0)|^2 dx = \int_{\mathbb{R}^N} h(t_0 \mathbf{w}_\infty(x, 0)) t_0^{-1} \mathbf{w}_\infty(x, 0) dx.$$

Since $\mathbf{w}_\infty \in \mathcal{M}_\infty$, it has to be $t_0 = 1$, proving our claim.

Then $I_\infty(t_R w_R) \rightarrow I_\infty(\mathbf{w}_\infty) = c(\mathbb{R}^N)$ as $R \rightarrow \infty$ and recalling (4.9),

$$(4.10) \quad \limsup_{\lambda \rightarrow \infty} c(\Omega_\lambda) \leq c(\mathbb{R}^N).$$

On the other hand, by the definition of $c(\Omega_\lambda)$ and $c(\mathbb{R}^N)$, we get $c(\Omega_\lambda) \geq c(\mathbb{R}^N)$ for all $\lambda > 0$, which implies

$$(4.11) \quad \liminf_{\lambda \rightarrow \infty} c(\Omega_\lambda) \geq c(\mathbb{R}^N).$$

The conclusion follows by (4.10) and (4.11). \square

Before to proceed, we need to introduce other notations. Given $a \in (-\infty, +\infty]$, we set

- $I_\lambda^a := \left\{ u \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : I_\lambda(u) \leq a \right\}$, the a -sublevel of I_λ ;
- $\mathcal{M}_\lambda^a := \mathcal{M}_\lambda \cap I_\lambda^a$.

Moreover, from now on we fix a real number $r > 0$ such that the sets

$$\Omega_\lambda^+ = \{x \in \mathbb{R}^N; d(x, \Omega_\lambda) \leq r\}$$

and

$$\Omega_\lambda^- = \{x \in \Omega; d(x, \partial\Omega_\lambda) \geq r\}$$

are homotopically equivalent to $\overline{\Omega}_\lambda$ and $B_{\lambda r} \subset \Omega_\lambda$, so that $\mathcal{M}_\lambda^{c(B_{\lambda r})} \neq \emptyset$.

The next proposition will be of primary importance in order to apply the ‘‘barycenter method’’.

Proposition 4.5. *There exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$,*

$$v \in \mathcal{M}_\lambda^{c(B_{\lambda r})} \implies \beta(v) \in \Omega_\lambda^+.$$

Proof. Suppose that there exist $\lambda_n \rightarrow \infty$, $v_n \in \mathcal{M}_{\lambda_n}^{c(B_{\lambda_n r})}$, that we may assume positive, such that

$$x_n = \beta(v_n^+(\cdot, 0)) \notin \Omega_{\lambda_n}^+.$$

Fixing $R_n > \text{diam}(\Omega_{\lambda_n})$, we have that

$$A_{\lambda_n R, \lambda_n r}(x_n) \supset \Omega_{\lambda_n}$$

and so, recalling (4.1)-(4.3),

$$(4.12) \quad a(R, r, \lambda_n) = a(R, r, \lambda_n, x_n) \leq I_{\lambda_n}(v_n) \leq c(B_{\lambda_n r}).$$

Sending $n \rightarrow \infty$ in (4.12) and using Proposition 4.4, it follows that

$$\limsup_{n \rightarrow \infty} a(R, r, \lambda_n) \leq c(\mathbb{R}^N)$$

which contradicts Proposition 4.2. \square

For $\lambda > 0$, we define the injective operator $\Psi_{\lambda, r} : \Omega_{\lambda}^- \rightarrow H_{0,L}^1(\mathcal{C}_{\Omega_{\lambda}}, y^{1-2\alpha})$ given, for every $\tilde{x} \in \Omega_{\lambda}^-$ by

$$[\Psi_{\lambda, r}(\tilde{x})](x, y) = \begin{cases} t_{\lambda} \mathfrak{w}_{B_{\lambda r}}(|\tilde{x} - x|, y) & \text{for } (x, y) \in \mathcal{C}_{B_{\lambda r}}(\tilde{x}) \\ 0 & \text{for } (x, y) \in \mathcal{C}_{\Omega_{\lambda} \setminus B_{\lambda r}}(\tilde{x}) \end{cases}$$

where $\mathfrak{w}_{B_{\lambda r}}$ is the ground state solution given in Proposition 4.3 and $t_{\lambda} > 0$ is such that $\Psi_{\lambda, r}(\tilde{x}) \in \mathcal{M}_{\lambda}$, see (2.6). Note that for every $\tilde{x} \in \Omega_{\lambda}^-$, it holds

$$\beta(\Psi_{\lambda, r}(\tilde{x})) = \beta([\Psi_{\lambda, r}(\tilde{x})](\cdot, 0)) = \tilde{x}$$

and since

$$I_{\lambda}(\Psi_{\lambda, r}(\tilde{x})) = I_{B_{\lambda r}}(t_{\lambda} \mathfrak{w}_{B_{\lambda r}}(|\tilde{x} - \cdot|, \cdot)) \leq I_{B_{\lambda r}}(\mathfrak{w}_{B_{\lambda r}}(|\tilde{x} - \cdot|, \cdot)) = c(B_{\lambda r}),$$

we infer also

$$\Psi_{\lambda, r}(\tilde{x}) \in \mathcal{M}_{\lambda}^{c(B_{\lambda r})}.$$

Then we have

Lemma 4.6. *For $\lambda \geq \lambda^*$ given in Proposition 4.5, the composite map*

$$\Omega_{\lambda}^- \xrightarrow{\Psi_{\lambda, r}} \mathcal{M}_{\lambda}^{c(B_{\lambda r})} \xrightarrow{\beta} \Omega_{\lambda}^+$$

is well defined and coincide with the inclusion map of Ω_{λ}^- into Ω_{λ}^+

The next result is a consequence of the above setting, but for the sake of completeness we give the proof. It is understood, from now on, that for λ^* we mean that given in Proposition 4.5.

Proposition 4.7. *For every $\lambda \geq \lambda^*$ we have*

$$\text{cat } \mathcal{M}_{\lambda}^{c(B_{\lambda r})} \geq \text{cat } \Omega_{\lambda}.$$

Proof. Assume that $\text{cat } \mathcal{M}_{\lambda}^{c(B_{\lambda r})} = n$. This means that n is the smallest positive integer such that

$$\mathcal{M}_{\lambda}^{c(B_{\lambda r})} = \bigcup_{j=1}^n A_j,$$

where $A_j, j = 1, \dots, n$ are closed and contractible in $\mathcal{M}_{\lambda}^{c(B_{\lambda r})}$; that is, there exist $h_j \in C([0, 1] \times A_j, \mathcal{M}_{\lambda}^{c(B_{\lambda r})})$ and fixed elements $w_j \in \mathcal{M}_{\lambda}^{c(B_{\lambda r})}$ such that

$$h_j(0, u) = u \text{ for all } u \in A_j \quad \text{and} \quad h_j(1, u) = w_j \text{ for all } u \in A_j.$$

Consider the closed sets $D_j = \Psi_{\lambda,r}^{-1}(A_j)$ and note that

$$\Omega_\lambda^- = \bigcup_{j=1}^n D_j.$$

Using the deformation $g_j : [0, 1] \times D_j \rightarrow \Omega_\lambda^+$ given by

$$g_j(t, x) = \beta\left((h_j(t, \Psi_{\lambda,r}(x)))^+(\cdot, 0)\right),$$

we have for $j = 1, \dots, n$ and $x \in D_j$

$$g_j(0, x) = \beta\left((h_j(0, \Psi_r(x)))^+(\cdot, 0)\right) = \beta\left(\Psi_{\lambda,r}(x)(\cdot, 0)\right) = x$$

and

$$g_j(1, x) = \beta\left((h_j(1, \Psi_r(z)))^+(\cdot, 0)\right) = \beta\left(w_j(\cdot, 0)^+\right) \in \Omega_\lambda^+.$$

This means that $D_j, j = 1, \dots, n$ is contractible in Ω_λ^+ , hence $\text{cat}_{\Omega_\lambda^+}(\Omega_\lambda^-) \leq n$. The conclusion follows since Ω_λ^+ and Ω_λ^- are homotopically equivalent to $\overline{\Omega}_\lambda$. \square

5. PROOF OF THEOREM 1.1 AND THEOREM 1.2

Let us fix $\lambda \geq \lambda^*$. Since I_λ satisfies the Palais-Smale condition on \mathcal{M}_λ , applying the Ljusternik-Schnirelmann theory and Proposition 4.7, we get I_λ on \mathcal{M}_λ has at least $\text{cat } \Omega_\lambda$ critical points whose energy is less than $c(B_{\lambda r})$. Moreover, all solutions obtained are positive by the maximum principle proved in Lemma 4.1, finishing the proof of Theorem 1.1.

To get another solution, and then proving Theorem 1.2, we use the same ideas of [10]. Since Ω_λ is not contractible, the compact set $A := \overline{\Phi_{\lambda,r}(\Omega_\lambda^-)}$ can not be contractible in $\mathcal{M}_\lambda^{c(B_{\lambda r})}$. Moreover, as in [9], one can show that functions on the Nehari manifold have to be positive on a set of nonzero measure.

In the following, for $u \in H_{0,L}^1(\Omega_\lambda, y^{1-2\alpha}) \setminus \{0\}$ we denote with $t_\lambda(u) > 0$ the unique positive number such that $t_\lambda(u)u \in \mathcal{M}_\lambda$.

Take $u^* \in H_{0,L}^1(\Omega_\lambda, y^{1-2\alpha})$ such that $u^* \geq 0$, and $I_\lambda(t_\lambda(u^*)u^*) > c(B_{\lambda r})$. Consider the cone

$$\mathcal{K} := \left\{ tu^* + (1-t)u : t \in [0, 1], u \in A \right\}$$

(which is compact and contractible) and, since functions in \mathcal{K} have to be positive on a set of nonzero measure, $0 \notin \mathcal{K}$. Then it makes sense to project the cone on the Nehari manifold

$$t_\lambda(\mathcal{K}) := \left\{ t_\lambda(w)w : w \in \mathcal{K} \right\} \subset \mathcal{M}_\lambda$$

and consider the number

$$c := \max_{t_\lambda(\mathcal{K})} I_\lambda > c(B_{\lambda r}).$$

Since $A \subset t_\lambda(\mathcal{K}) \subset \mathcal{M}_\lambda$ and $t_\lambda(\mathcal{K})$ is contractible in \mathcal{M}_λ^c , we infer that also A is contractible in \mathcal{M}_λ^c . In conclusion, A is contractible in \mathcal{M}_λ^c , not contractible in $\mathcal{M}_\lambda^{c(B_{\lambda r})}$, and $c > c(B_{\lambda r})$; this is only possible, since I_λ satisfies the Palais-Smale condition, if there is a critical level between $c(B_{\lambda r})$ and c , that is, another solution to our problem.

6. PROOF OF THEOREM 1.3

Before prove the theorem we recall some basic facts of Morse theory and fix some notations. For a pair of topological spaces (X, Y) , $Y \subset X$, let $H_*(X, Y)$ be its singular homology with coefficients in some field \mathbb{F} (from now on omitted) and

$$\mathcal{P}_t(X, Y) = \sum_k \dim H_k(X, Y) t^k$$

the Poincaré polynomial of the pair. If $Y = \emptyset$, it will be always omitted in the objects which involve the pair. Recall that if H is an Hilbert space, $I : H \rightarrow \mathbb{R}$ a C^2 functional and u an isolated critical point with $I(u) = c$, the *polynomial Morse index* of u is

$$\mathcal{I}_t(u) = \sum_k \dim C_k(I, u) t^k$$

where $C_k(I, u) = H_k(I^c \cap U, (I^c \setminus \{u\}) \cap U)$ are the critical groups. Here $I^c = \{u \in H : I(u) \leq c\}$ and U is a neighborhood of the critical point u . The multiplicity of u is the number $\mathcal{I}_1(u)$.

It is known that for a non-degenerate critical point u (that is, the selfadjoint operator associated to $I''(u)$ is an isomorphism) it is $\mathcal{I}_t(u) = t^{\mathfrak{m}(u)}$, where $\mathfrak{m}(u)$ is the (*numerical*) *Morse index of u* : the maximal dimension of the subspaces where $I''(u)[\cdot, \cdot]$ is negative definite.

Coming back to our functional, we know that I_λ satisfies the Palais-Smale condition (see Lemma 3.2). Moreover I_λ is of class C^2 and for $v, v_1, v_2 \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ it is

$$\begin{aligned} I_\lambda''(v)[v_1, v_2] &= k_\alpha^{-1} \int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha} \nabla v_1 \nabla v_2 dx dy + \\ &\quad \int_{\Omega_\lambda} v(x, 0) w(x, 0) dx - \int_{\Omega_\lambda} h'(v(x, 0)) v_1(x, 0) v_2(x, 0) dx. \end{aligned}$$

So $I_\lambda''(v)$ is represented by the operator

$$(6.1) \quad L_\lambda(v) := R_\lambda(v) - K_\lambda(v) : H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \rightarrow \left(H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \right)'$$

where $R_\lambda(v)$ is the Riesz isomorphism and $K_\lambda(v)$ is compact. Indeed let $v_n \rightharpoonup 0$ in $H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ and $w \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$; in virtue of (H1') and (H2'), for a given $\xi > 0$ there exists some constant $C_\xi > 0$ such that

$$\int_{\Omega_\lambda} \left| h'(v(x, 0)) v_n(x, 0) w(x, 0) \right| dx \leq \xi \int_{\Omega_\lambda} |v_n(x, 0) w(x, 0)| dx + C_\xi \int_{\Omega_\lambda} |v(x, 0)|^{q-1} |v_n(x, 0) w(x, 0)| dx.$$

Using that $v_n \rightharpoonup 0$ and the arbitrariness of ξ , we get

$$\|K_\lambda(v)[v_n]\| = \sup_{\|w\|_\alpha=1} \left| \int_{\Omega_\lambda} h'(v(x, 0)) v_n(x, 0) w(x, 0) dx \right| \rightarrow 0.$$

In particular $L_\lambda(v)$ is a Fredholm operator with index zero. Moreover, for $a \in (-\infty, +\infty]$, we set

- $\text{Crit}_\lambda := \left\{ u \in H_{0,L}^1(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : I'_\lambda(u) = 0 \right\}$, the set of critical points of I_λ ;
- $(\text{Crit}_\lambda)^a := \text{Crit}_\lambda \cap I_\lambda^a$;
- $(\text{Crit}_\lambda)_a := \left\{ u \in \text{Crit}_\lambda : I_\lambda(u) > a \right\}$.

In the remaining part of this section we will follow [6, 9]. We will not give the proofs of the next Lemma 6.1 and Corollary 6.2 since they follow by general arguments.

Let $\lambda^* > 0$ as given in Proposition 4.5 and $\lambda \geq \lambda^*$ be fixed from now on. In view of Corollary 2.2, to prove Theorem 1.3 it is sufficient to show that I_λ restricted to \mathcal{M}_λ has at least $2\mathcal{P}_1(\Omega_\lambda) - 1$ critical points.

First note that we can assume that $c(B_{\lambda r})$ is a regular value for I_λ . Otherwise we can choose a $\rho \in (0, r)$ so that the new sets

$$\Omega_\lambda^+ = \{x \in \mathbb{R}^N; d(x, \Omega_\lambda) \leq \rho\} \quad \text{and} \quad \Omega_\lambda^- = \{x \in \Omega; d(x, \partial\Omega_\lambda) \geq \rho\}$$

are still homotopically equivalent to Ω , $c(B_{\lambda\rho}) > c(B_{\lambda r})$ and $c(B_{\lambda\rho})$ is a regular value; and we rename $c(B_{\lambda\rho})$ as $c(B_{\lambda r})$. Of course, we can also assume that Crit_λ is discrete. Since I_λ is bounded from below on \mathcal{M}_λ , let us say by a $\delta_\lambda > 0$, we have

$$(\text{Crit}_\lambda)^{c(B_{\lambda r})} = \left\{ v \in \text{Crit}_\lambda : 0 < \delta_\lambda < I_\lambda(v) \leq c(B_{\lambda r}) \right\}$$

and $(\text{Crit}_\lambda)^{c(B_{\lambda r})}$ and $(\text{Crit}_\lambda)_{c(B_{\lambda r})}$ are (critical) isolated sets covering Crit_λ .

By Lemma 4.6 and the fact that $(\Psi_{\lambda, r})_*$ induces monomorphism between the homology groups $H_*(\Omega_\lambda^-)$ and $H_*(\mathcal{M}_\lambda^{c(B_{\lambda r})})$, it is standard to see that

$$(6.2) \quad \mathcal{P}_t(\mathcal{M}_\lambda^{c(B_{\lambda r})}) = \mathcal{P}_t(\Omega_\lambda^-) + \mathcal{Q}_t, \quad \mathcal{Q} \in \mathbb{P}$$

where we are denoting with \mathbb{P} the set of polynomial with nonnegative integer coefficients. Recall that $c(\Omega_\lambda) = \min_{\mathcal{M}_\lambda} I_\lambda$. As in [9, Lemma 5.2] (the proof just uses a topological lemma and a general deformation argument) one proves the following

Lemma 6.1. *Let $d \in (0, c(\Omega_\lambda))$ and $l \in (d, +\infty]$ a regular level for I_λ . Then*

$$\mathcal{P}_t(I_\lambda^l, I_\lambda^d) = t\mathcal{P}_t(\mathcal{M}_\lambda^l).$$

From this lemma, (6.2) and the fact that $\pi_1(\mathcal{M}_\lambda) \approx \{0\}$, it follows that

$$(6.3) \quad \mathcal{P}_t(I_\lambda^{c(B_{\lambda r})}, I_\lambda^d) = t\left(\mathcal{P}_t(\Omega_\lambda^-) + \mathcal{Q}_t\right)$$

and

$$(6.4) \quad \mathcal{P}_t(H_{0,L}^1(y^{1-2\alpha}), I_\lambda^d) = t\mathcal{P}_t(\mathcal{M}_\lambda) = t.$$

Finally we need the next result, whose proof is a matter of algebraic topology (see [6, Lemma 2.4] or [9, Lemma 5.6])

Corollary 6.2. *We have*

$$(6.5) \quad \mathcal{P}_t(H_{0,L}^1(y^{1-2\alpha}), I_\lambda^{c(B_{\lambda r})}) = t^2\left(\mathcal{P}_t(\Omega_\lambda) + \mathcal{Q}_t - 1\right), \quad \mathcal{Q} \in \mathbb{P}.$$

Then the Morse theory, (6.3), (6.4) and (6.5) give

$$\begin{aligned} \sum_{v \in (\text{Crit}_\lambda)^{c(B_{\lambda r})}} \mathcal{I}_t(v) &= \mathcal{P}_t(I_\lambda^{c(B_{\lambda r})}, I_\lambda^d) + (1+t)\mathcal{Q}'_t \\ &= t\left(\mathcal{P}_t(\Omega_\lambda) + \mathcal{Q}_t\right) + (1+t)\mathcal{Q}'_t \end{aligned}$$

and

$$\begin{aligned} \sum_{v \in (\text{Crit}_\lambda)_{c(B_{\lambda r})}} \mathcal{I}_t(v) &= \mathcal{P}_t(H_{0,L}^1(y^{1-2\alpha}), I_\lambda^{c(B_{\lambda r})}) + (1+t)\mathcal{Q}''_t \\ &= t^2\left(\mathcal{P}_t(\Omega_\lambda) + \mathcal{Q}_t - 1\right) + (1+t)\mathcal{Q}''_t \end{aligned}$$

for some $\mathcal{Q}, \mathcal{Q}', \mathcal{Q}'' \in \mathbb{P}$. As a consequence we obtain

$$(6.6) \quad \sum_{v \in \text{Crit}_\lambda} \mathcal{I}_t(v) = t\mathcal{P}_t(\Omega_\lambda) + t^2(\mathcal{P}_t(\Omega_\lambda) - 1) + t(1+t)\mathcal{Q}_t$$

for a suitable $\mathcal{Q} \in \mathbb{P}$.

It is known that for a non-degenerate critical point v (that is, $L_\lambda(v)$ given in (6.1) is an isomorphism) it is $\mathcal{I}_t(v) = t^{\mathfrak{m}(v)}$, where $\mathfrak{m}(v)$ is the (numerical) Morse index of v : the maximal dimension of the subspaces where $I''_\lambda(v)[\cdot, \cdot]$ is negative definite.

Then, if the solutions are non-degenerate, (6.6) easily gives the existence of at least $2\mathcal{P}_1(\Omega_\lambda) - 1$ solutions, completing the proof of Theorem 1.3.

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